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The t -intersection Problem in the Truncated Boolean Lattice

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Let $I(n, t)$ be the class of all t -intersecting families of subsets of $[n]$ and set $I_k(n, t) = I(n, t) \cap 2^{\binom{[n]}{\leq k}}$, $I_{\leq k}(n, t) = I(n, t) \cap 2^{\binom{[n]}{\leq k}}$.

After the maximal families in $I(n, t)$ [13] and in $I_k(n, t)$ [1, 9] are known we study now maximal families in $I_{\leq k}(n, t)$. We present a conjecture about the maximal cardinalities and prove it in several cases.

More generally cardinalities are replaced by weights and asymptotic estimates are given.

Analogous investigations are made for $I(n, t) \cap C(n, s)$, where $C(n, s)$ is the class of all s -cointersecting families of subsets of $[n]$. In particular we establish an asymptotic form of a conjecture by Bang *et al.* [4].

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1. INTRODUCTION AND NOTATION

Let \mathbb{N} be the set of natural numbers, $[n] := \{1, \dots, n\}$, and for $i, j \in \mathbb{N}$, $i < j$, let $[i, j] := \{i, i+1, \dots, j\}$. Let $2^{[n]}$ be the family of all subsets of $[n]$. Also, let

$$\binom{[n]}{k} := \{X \subseteq [n] : |X| = k\}, \quad \binom{[n]}{\leq k} := \{X \subseteq [n] : |X| \leq k\},$$

$$\binom{[n]}{\geq k} := \{X \subseteq [n] : |X| \geq k\}.$$

A family $\mathcal{F} \subseteq 2^{[n]}$ is called t -intersecting (resp. s -cointersecting) if, for all $X, Y \in \mathcal{F}$, $|X \cap Y| \geq t$ (resp. $|X \cup Y| \leq n - s$). Let $I(n, t)$ (resp. $C(n, s)$) be the class of all t -intersecting (resp. s -cointersecting) families of subsets of $[n]$. Furthermore, let

$$I_k(n, t) := I(n, t) \cap 2^{\binom{[n]}{k}}, \quad I_{\leq k}(n, t) := I(n, t) \cap 2^{\binom{[n]}{\leq k}},$$

i.e., the class of t -intersecting families whose members have size equal to k resp. not greater than k , and let $I_{\geq k}(n, t)$, $C_{\leq k}(n, s)$, $C_{\geq k}(n, s)$ be defined analogously.

For a class \mathcal{K} of families, let

$$M(\mathcal{K}) := \max\{|\mathcal{F}| : \mathcal{F} \in \mathcal{K}\}.$$

More generally, if there is given a weight function $\omega : 2^{[n]} \rightarrow \mathbb{R}_+$ (the set of all nonnegative reals), let for $\mathcal{F} \subseteq 2^{[n]}$

$$\omega(\mathcal{F}) := \sum_{X \in \mathcal{F}} \omega(X)$$

and

$$M(\mathcal{K}, \omega) := \max\{\omega(\mathcal{F}) : \mathcal{F} \in \mathcal{K}\}.$$

In this paper we study the numbers $M(\mathcal{K})$ for

$$\mathcal{K} \in \{I_{\leq k}(n, t), I_{\geq k}(n, t), C_{\leq k}(n, s), C_{\geq k}(n, s), I(n, t) \cap C(n, s)\}.$$

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2. RESULTS

First of all, by considering complements

$$\begin{aligned} M(C_{\geq k}(n, s)) &= M(I_{\leq n-k}(n, s)), \\ M(C_{\leq k}(n, s)) &= M(I_{\geq n-k}(n, s)), \end{aligned}$$

so that only three of the five numbers are of interest.

Let, for $r = 0, \dots, \lfloor \frac{n-t}{2} \rfloor$,

$$\begin{aligned} S(n, t, r) &:= \{X \in 2^{[n]} : |X \cap [t + 2r]| \geq t + r\}, \\ S_k(n, t, r) &:= S(n, t, r) \cap \binom{[n]}{k}, \\ S_{\leq k}(n, t, r) &:= S(n, t, r) \cap \binom{[n]}{\leq k}, \end{aligned}$$

and let $S_{\geq k}(n, t, r)$ be defined analogously. By construction, these families are t -intersecting.

The following two results are basic for our investigation.

THEOREM 1 (KATONA [13]). *We have*

$$M(I(n, t)) = \left| S\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right) \right|.$$

THEOREM 2 (AHLWEDE, KHACHATRIAN [1]). *We have*

$$M(I_k(n, t)) = \max \left\{ |S_k(n, t, r)| : r = 0, \dots, \left\lfloor \frac{n-t}{2} \right\rfloor \right\}.$$

Moreover, for $n > 2k - t$, the optimal r is given by

$$\frac{(k-t+1)(t-1)}{n-2k+2t-2} - 1 \leq r \leq \frac{(k-t+1)(t-1)}{n-2k+2t-2}.$$

An easy consequence of Theorem 1 is the following (cf. [6, 8]):

THEOREM 3. *Let $\omega(X) = \omega(Y)$ for all $X, Y \subseteq [n]$ with $|X| = |Y|$ and let $\omega(X) \leq \omega(Y)$ if $|X| + |Y| = n + t - 1$, $|X| \leq |Y|$. Then*

$$M(I(n, t), \omega) = \omega\left(S\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right)\right).$$

Setting

$$\omega(X) := \begin{cases} 1 & \text{if } |X| \geq k \\ 0 & \text{otherwise} \end{cases}$$

we obtain immediately from Theorem 3:

COROLLARY 4. *We have*

$$M(I_{\geq k}(n, t)) = \left| S_{\geq k}\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right) \right|.$$

The determination of $M(I_{\leq k}(n, t))$ is more difficult and, up to now, we can provide only partial results.

PROPOSITION 5. *We have*

$$M(I_{\leq k}(n, 1)) = |S_{\geq k}(n, t, 0)|.$$

Indeed, this follows easily using complements and the Erdős–Ko–Rado theorem [9]. Hence we suppose throughout $t \geq 2$ when studying $I_{\leq k}(n, t)$.

The following question was the starting point of our investigations:

PROBLEM 6. For which numbers k do we have

$$M(I_{\leq k}(n, t)) = \left| S_{\leq k} \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right|? \quad (1)$$

Concerning this question we may clearly suppose that $k \geq \lfloor \frac{n+t}{2} \rfloor$ because otherwise $S_{\leq k}(n, t, \lfloor \frac{n-t}{2} \rfloor) = \emptyset$. Problem 6 is answered essentially by the following results:

THEOREM 7. *Let t and c be fixed constants and let $k \leq \frac{n+t}{2} + c\sqrt{n}$. Then (1) does not hold if n is large enough.*

THEOREM 8. *Let t be fixed and $k \geq \frac{n+t}{2} + \sqrt{\log n}\sqrt{n}$. Then (1) holds if n is large enough.*

THEOREM 9. *Let c be fixed constant and let $k \leq \frac{n+t}{2} + c$. Then there exists $\delta > 0$ such that for $t \leq \delta n$ and n sufficiently large (1) does not hold.*

THEOREM 10. *Let $\delta > 0$ be fixed constant and let $t \geq \delta n$. Then there exists $c > 0$ such that for $k \geq \frac{n+t}{2} + c$ and n sufficiently large (1) holds.*

Concerning the complete determination of $M(I_{\leq k}(n, t))$ we have the following conjecture:

CONJECTURE 11. *If $k < \frac{n+t}{2}$, then*

$$M(I_{\leq k}(n, t)) = \max \left\{ |S_{\leq k}(n, t, r)| : r = 0, \dots, \left\lfloor \frac{n-t}{2} \right\rfloor \right\}. \quad (2)$$

This conjecture is supported by the following results.

THEOREM 12. *Let t and $0 < \epsilon < \frac{1}{2}$ be fixed constants and $k \leq (\frac{1}{2} - \epsilon)n$. Then (2) holds for sufficiently large n .*

THEOREM 13. *Let $t = \tau n + o(n)$ and $k = \kappa n + o(n)$ with $0 < \tau < \kappa < \frac{1+\tau}{2}$. Then, as $n \rightarrow \infty$*

$$M(I_{\leq k}(n, t)) \sim \max \left\{ |S_{\leq k}(n, t, r)| : r = 0, \dots, \left\lfloor \frac{n-t}{2} \right\rfloor \right\}.$$

Studying $M(I(n, t) \cap C(n, s))$ one can clearly suppose throughout that $t + s \leq n$. Given n , t , s and $r \in \{0, \dots, \lfloor \frac{n-t-s}{2} \rfloor\}$, always let

$$q := \left\lfloor \frac{n-t-s}{2} \right\rfloor - r.$$

Note that

$$(t + 2r) + (s + 2q) = \begin{cases} n & \text{if } 2 \mid n - s - t \\ n - 1 & \text{otherwise.} \end{cases}$$

Let, for $r = 0, \dots, \lfloor \frac{n-t-s}{2} \rfloor$,

$$S(n, t, s, r) := \{X \in 2^{[n]} : |X \cap [t + 2r]| \geq t + r \text{ and } |X \cap [n - s - 2q + 1, n]| \leq q\}.$$

Obviously, these families are t -intersecting and s -cointersecting. Verifying a conjecture of Katona, Frankl [10] proved:

THEOREM 14. *We have*

$$M(I(n, 1) \cap C(n, s)) = |S(n, 1, s, 0)|.$$

Moreover, Frankl [11] and Bang *et al.* [4] propose:

CONJECTURE 15. *We have*

$$M(I(n, t) \cap C(n, s)) = \max \left\{ |S(n, t, s, r)| : r = 0, \dots, \left\lfloor \frac{n-t-s}{2} \right\rfloor \right\}.$$

In [4] this conjecture is proved for $n-t-s \leq 3$.

From Theorem 1 one easily obtains that for fixed t

$$M(I(n, t)) \sim 2^{n-1} \quad \text{as } n \rightarrow \infty.$$

This gives, applying in a standard way Kleitman's inequality (cf. [7, p. 266]):

PROPOSITION 16. *Let t and s be fixed and let $n \rightarrow \infty$. Then*

$$M(I(n, t) \cap C(n, s)) \sim 2^{n-2} \sim \max \left\{ |S(n, t, s, r)| : r = 0, \dots, \left\lfloor \frac{n-t-s}{2} \right\rfloor \right\}.$$

In addition, we have the following result:

THEOREM 17. *Let $t = \tau n + o(n)$, $s = \sigma n + o(n)$, $\tau, \sigma > 0$, $\tau + \sigma < 1$ and $n \rightarrow \infty$. Then*

$$M(I(n, t) \cap C(n, s)) \sim \max \left\{ |S(n, t, s, r)| : r = 0, \dots, \left\lfloor \frac{n-t-s}{2} \right\rfloor \right\}.$$

Thus Conjecture 15 is supported by Proposition 16 and Theorem 17.

3. SHORT PROOFS FOR RESULTS CONCERNING $I_{\leq k}(n, t)$

PROOF OF THEOREM 7. It is easy to see that (1) holds for some k if it holds for some k' with $k' < k$ (see Lemma 19). Hence it is sufficient to prove the assertion for

$$k = \left\lceil \frac{n+t}{2} + c\sqrt{n} \right\rceil.$$

We use the well-known fact that for constants a, b (with $a < b$) and for $n \rightarrow \infty$

$$\sum_{\frac{n}{2} + \frac{1}{2}\sqrt{na} + o(\sqrt{n}) \leq j \leq \frac{n}{2} + \frac{1}{2}\sqrt{nb} + o(\sqrt{n})} \binom{n}{j} \sim (\Phi(b) - \Phi(a))2^n \quad (3)$$

uniformly in $a, b \in \mathbb{R}$, where Φ is the Gaussian distribution. Since

$$\sum_{i=\lfloor \frac{n+t}{2} \rfloor + 1}^k \binom{n}{i} \leq \left| S_{\leq k} \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right| \leq \sum_{i=\lfloor \frac{n+t}{2} \rfloor}^k \binom{n}{i}$$

we have

$$\left| S_{\leq k} \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right| \sim (\Phi(2c) - \Phi(0))2^n = \left(\Phi(2c) - \frac{1}{2} \right) 2^n. \quad (4)$$

Now choose $r := \lfloor n^{\frac{1}{4}} \rfloor$. From (3) it follows that

$$\sum_{j=0}^{k-i} \binom{n-t-2r}{j} \sim \Phi(2c)2^{n-t-2r}$$

uniformly in $i \in [t+r, t+2r]$ and that

$$\sum_{i=t+r}^{t+2r} \binom{t+2r}{i} \sim \Phi(0)2^{t+2r}.$$

Consequently,

$$\begin{aligned} |S_{\leq k}(n, t, r)| &= \sum_{i=t+r}^{t+2r} \binom{t+2r}{i} \sum_{j=0}^{k-i} \binom{n-t-2r}{j} \\ &\sim \Phi(0)2^{t+2r} \Phi(2c)2^{n-t-2r} = \frac{1}{2} \Phi(2c)2^n. \end{aligned} \quad (5)$$

Since $\Phi(2c) - \frac{1}{2} < \frac{1}{2} \Phi(2c)$ we have by (4) and (5) for sufficiently large n ,

$$\left| S_{\leq k} \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right| < |S_{\leq k}(n, t, r)|. \quad \square$$

PROOF OF THEOREM 9. Analogously to the proof of Theorem 7 we prove the assertion only for

$$k = \left\lceil \frac{n+t}{2} + c \right\rceil.$$

W.l.o.g. we may assume that c is an integer. Moreover, we suppose that $2 \mid n+t$. If $2 \nmid n+t$ the proof can be modified in a straightforward way. We have $k = \frac{n+t}{2} + c$ and put $d := 3(c+2)^2$. Note that for constant integers a and b

$$\frac{\binom{n-a}{\ell}}{\binom{n}{\ell+b}} \sim (1 - \ell/n)^a \left(\frac{\ell/n}{1 - \ell/n} \right)^b. \quad (6)$$

Let $\tau := \frac{t}{n}$. We take $r := \frac{n-t}{2} - d$ and compare $|S_{\leq k}(n, t, r)|$ with $|S_{\leq k}(n, t, \lfloor \frac{n-t}{2} \rfloor)|$. We have (with $t+r+c+d=k$)

$$|S_{\leq k}(n, t, r)| = \sum_{i=0}^{c+d} \binom{n-2d}{t+r+i} \sum_{j=0}^{c+d-i} \binom{2d}{j}.$$

Using (6) we obtain

$$\frac{|S_{\leq k}(n, t, r)|}{\binom{n}{(n+t)/2}} \sim \left(\frac{1-\tau}{2} \right)^{2d} \sum_{i=0}^{c+d} \left(\frac{1+\tau}{1-\tau} \right)^{d-i} \sum_{j=0}^{c+d-i} \binom{2d}{j}.$$

Analogously,

$$\begin{aligned} \left| S_{\leq k} \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right| &= \sum_{j=0}^c \binom{n}{(n+t)/2 + j}, \\ \frac{|S_{\leq k}(n, t, \lfloor \frac{n-t}{2} \rfloor)|}{\binom{n}{(n+t)/2}} &\sim \sum_{j=0}^c \left(\frac{1+\tau}{1-\tau} \right)^{-j}. \end{aligned}$$

For the proof it is enough to show that there are $\epsilon, \delta > 0$ such that for $\tau \leq \sigma$, independently of n ,

$$\left(\frac{1-\tau}{2}\right)^{2d} \sum_{i=0}^{c+d} \left(\frac{1+\tau}{1-\tau}\right)^{d-i} \sum_{j=0}^{c+d-i} \binom{2d}{j} \geq \sum_{j=0}^c \left(\frac{1+\tau}{1-\tau}\right)^{-j} + \epsilon \quad (7)$$

since then for sufficiently large n and $t \leq \tau n$

$$|S_{\leq k}(n, t, r)| > \left| S_{\leq k}\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right) \right|.$$

Both sides of (7) are continuous functions of τ . Hence it is enough to consider $\tau = 0$ and to prove

$$L := \sum_{i=0}^{c+d} \sum_{j=0}^{c+d-i} \binom{2d}{j} > (c+1)2^{2d} =: R. \quad (8)$$

Let $a \in \{0, \dots, c-1\}$ and consider on the LHS of (8) the terms with $i = a$ and $i = 2c - a$. We have

$$\begin{aligned} \sum_{j=0}^{c+d-a} \binom{2d}{j} + \sum_{j=0}^{c+d-(2c-a)} \binom{2d}{j} &= \sum_{j=0}^{c+d-a} \binom{2d}{j} + \sum_{j=0}^{d-c+a} \binom{2d}{2d-j} \\ &= \sum_{j=0}^{c+d-a} \binom{2d}{j} + \sum_{j=c+d-a}^{2d} \binom{2d}{j} \\ &> 2^{2d}. \end{aligned}$$

For $i = c$,

$$\sum_{j=0}^{c+d-i} \binom{2d}{j} = \frac{1}{2}2^{2d} + \frac{1}{2}\binom{2d}{d}.$$

Consequently, we have the following estimation for the LHS of (8):

$$L > \left(c + \frac{1}{2}\right)2^{2d} + \frac{1}{2}\binom{2d}{d} + \sum_{i=2c+1}^{c+d} \sum_{j=0}^{c+d-i} \binom{2d}{j}. \quad (9)$$

For $i \geq 2c+1$,

$$\begin{aligned} \sum_{j=0}^{c+d-i} \binom{2d}{j} &= \sum_{j=0}^d \binom{2d}{j} - \sum_{j=c+d-i+1}^d \binom{2d}{j} \\ &> \frac{1}{2}2^{2d} + \frac{1}{2}\binom{2d}{d} - (i-c)\binom{2d}{d} \\ &= \frac{1}{2}2^{2d} - \left(i - c - \frac{1}{2}\right)\binom{2d}{d}. \end{aligned}$$

Considering in (8) only the terms with $i = 2c+1, 2c+2, 2c+3$ gives

$$L > (c+1)2^{2d} + 2^{2d} - (3c+4)\binom{2d}{d}.$$

Accordingly, $L > R$ (i.e., (7) holds) if

$$2^{2d} > \binom{2d}{d}(3c+4). \quad (10)$$

It is well-known (cf. [12, p. 283]) that

$$\binom{2d}{d} \leq \frac{2^{2d}}{\sqrt{3d+1}}.$$

Hence (10) holds if $\sqrt{3d+1} > 3c+4$. Indeed (using $d = 3(c+2)^2$), $\sqrt{3d+1} > \sqrt{9(c+2)^2} = 3(c+2) > 3c+4$. \square

4. ASYMPTOTIC ESTIMATES OF $M(I_{\leq k}(n, t))$ AND $M(I(n, t) \cap C(n, s))$

PROOF OF THEOREM 13. For any family \mathcal{F} we use the notation

$$\mathcal{F}_h := \{X \in \mathcal{F} : |X| = h\}.$$

Let $\mathcal{F} \in I_{\leq k}(n, t)$. Clearly,

$$|\mathcal{F}| = \sum_{h=0}^k |\mathcal{F}_h|. \quad (11)$$

First we estimate each $|\mathcal{F}_h|$. In the following the maximum is always extended over $r \in \{0, \dots, \lfloor \frac{n-t}{2} \rfloor\}$. By Theorem 2,

$$\begin{aligned} |\mathcal{F}_h| &\leq \max\{|S_h(n, t, r)|\} = \max \left\{ \sum_{i=0}^r \binom{t+2r}{r-i} \binom{n-t-2r}{h-t-r-i} \right\} \\ &\leq \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{h-t-r} \sum_{i=0}^{\infty} \left(\frac{r}{t+r+1} \frac{h-t-r}{n-h-r+1} \right)^i \right\} \\ &\leq \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{k-t-r} \left(\frac{k-t-r}{n-k-r+1} \right)^{k-h} \frac{1}{1 - \frac{r}{t+r+1} \frac{k-t-r}{n-k-r+1}} \right\}. \quad (12) \end{aligned}$$

We will see that almost all numbers $|\mathcal{F}_h|$ can be neglected. Only the values $|\mathcal{F}_h|$ with h near to k give an essential contribution. Clearly, it is enough to extend the maximum only over $r \in \{0, \dots, k-t\}$. Then

$$\frac{r}{t+r+1} \leq \frac{k-t}{k+1} = 1 - \frac{\tau}{\kappa} + o(1).$$

Moreover, for large n , $k-t-r < n-k-r+1$, hence

$$\frac{k-t-r}{n-k-r+1} \leq \frac{k-t}{n-k+1} = \frac{\kappa-\tau}{1-\kappa} + o(1) < 1.$$

Choose α such that $\frac{\kappa-\tau}{1-\kappa} < \alpha < 1$. Then, for any $\epsilon > 0$ and any h with $h \leq k - \epsilon n$,

$$|\mathcal{F}_h| \leq \frac{1}{(1-\tau/\kappa)\alpha} \alpha^{\epsilon n} \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{k-t-r} \right\}$$

and

$$\sum_{h \leq k - \epsilon n} |\mathcal{F}_h| \leq \frac{1}{(1-\tau/\kappa)\alpha} n \alpha^{\epsilon n} \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{k-t-r} \right\}. \quad (13)$$

We put $\epsilon := n^{-\frac{1}{2}}$. Now let h be near to k , i.e., $k - h \leq \epsilon n$. By Theorem 2, $\max\{|S_h(n, t, r)|\}$ is attained at some $r = r(k)$ with

$$\frac{(\kappa - \epsilon - \tau)\tau n}{1 - 2\kappa + 2\epsilon + 2\tau} - o(n) \leq r \leq \frac{(\kappa - \tau)\tau n}{1 - 2\kappa + 2\tau} + o(n).$$

Then, uniformly for $k - \epsilon n \leq h \leq k$,

$$\begin{aligned} \frac{r}{t + r + 1} &= \frac{\kappa - \tau}{1 - (\kappa - \tau)} + o(1), \\ \frac{k - t - r}{n - k - r + 1} &= \frac{\kappa - \tau}{1 - (\kappa - \tau)} + o(1). \end{aligned}$$

Let $\omega := \frac{\kappa - \tau}{1 - (\kappa - \tau)}$. From (12) we obtain

$$|\mathcal{F}_h| \leq \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{k-t-r} (\omega + o(1))^{k-h} \frac{1}{1 - \omega^2 - o(1)} \right\}$$

and, consequently,

$$\sum_{k-\epsilon n < h \leq k} |\mathcal{F}_h| \leq \frac{1}{1 - \omega} \frac{1}{1 - \omega^2} (1 + o(1)) \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{k-t-r} \right\}. \quad (14)$$

Since $n\alpha^{\epsilon n} = o(1)$, we finally get from (11), (13) and (14)

$$|\mathcal{F}| \leq \frac{1}{1 - \omega} \frac{1}{1 - \omega^2} (1 + o(1)) \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{k-t-r} \right\}. \quad (15)$$

On the other hand, using more or less the same estimations, one can derive $\max\{|S_{\leq k}(n, t, r)|\} \geq \frac{1}{1-\omega} \frac{1}{1-\omega^2} (1 + o(1)) \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{k-t-r} \right\}$ which proves together with (15) the assertion. \square

PROOF OF THEOREM 17. Let $\mathcal{F} \in I(n, t) \cap C(n, s)$. First let $2 \mid n + t + s$ and let $k := \frac{n+t-s}{2}$. We divide \mathcal{F} into two subfamilies

$$\mathcal{F}' := \bigcup_{h=0}^k \mathcal{F}_h, \quad \mathcal{F}'' := \bigcup_{h=k+1}^n \mathcal{F}_h$$

and put

$$\mathcal{F}''' := \{[n] \setminus X : X \in \mathcal{F}''\}.$$

Obviously, $\mathcal{F}' \in I_{\leq k}(n, t)$, $\mathcal{F}''' \in I_{\leq n-k-1}(n, s)$. Using the notations from Theorem 13 we have (for \mathcal{F}' and \mathcal{F}''')

$$\omega = \frac{1 - \tau - \sigma}{1 + \tau + \sigma}$$

and get the estimations

$$\begin{aligned} |\mathcal{F}'| &\leq \frac{1}{1 - \omega} \frac{1}{1 - \omega^2} (1 + o(1)) \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{(n-t-s)/2-r} \right\}, \\ |\mathcal{F}'''| &\leq \frac{1}{1 - \omega} \frac{1}{1 - \omega^2} (1 + o(1)) \max \left\{ \binom{s+2q}{q} \binom{n-s-2q}{(n-t-s)/2-q} \right\}, \\ &= \frac{\omega}{1 - \omega} \frac{1}{1 - \omega^2} (1 + o(1)) \max \left\{ \binom{s+2q}{q} \binom{n-s-2q}{(n-t-s)/2-q} \right\}, \end{aligned}$$

and, with $r := \frac{n-t-s}{2} - q$,

$$|\mathcal{F}'''| \leq \frac{\omega}{1-\omega} \frac{1}{1-\omega^2} (1+o(1)) \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{(n-t-s)/2-r} \right\}.$$

Consequently,

$$|\mathcal{F}| = |\mathcal{F}'| + |\mathcal{F}'''| \leq \frac{1}{(1-\omega)^2} (1+o(1)) \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{(n-t-s)/2-r} \right\}.$$

Again, in a similar way, one can derive that

$$\begin{aligned} \max \left\{ |S(n, t, s, r)| : r = 0, \dots, \frac{n-t-s}{2} \right\} \\ \geq \frac{1}{(1-\omega)^2} (1+o(1)) \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{(n-t-s)/2-r} \right\} \end{aligned}$$

which proves the assertion.

Now let $2 \nmid n+t+s$. Here we put $k := \frac{n+t-s-1}{2}$. With the same approach we get

$$\begin{aligned} |\mathcal{F}'| &\leq \frac{1}{1-\omega} \frac{1}{1-\omega^2} (1+o(1)) \max \left\{ \binom{t+2r}{r} \binom{n-t-2r}{(n-t-s-1)/2-r} \right\}, \\ |\mathcal{F}'''| &\leq \frac{1}{1-\omega} \frac{1}{1-\omega^2} (1+o(1)) \max \left\{ \binom{s+2q}{q} \binom{n-s-2q}{(n-t-s-1)/2-q} \right\}, \\ &= \frac{1}{1-\omega} \frac{1}{1-\omega^2} (1+o(1)) \max \left\{ \binom{t+2r+1}{r} \binom{n-1-t-2r}{(n-t-s-1)/2-r} \right\}. \end{aligned}$$

It is not difficult to verify that the maximum on both RHS is attained at some r with

$$r \sim \frac{\tau}{2} \frac{1-\tau-\sigma}{\tau+\sigma} n.$$

This easily implies

$$|\mathcal{F}| = |\mathcal{F}'| + |\mathcal{F}'''| \leq \frac{2}{(1-\omega)^2} (1+o(1)) \max \left\{ \binom{t+2r}{r} \binom{n-1-t-2r}{(n-t-s-1)/2-r} \right\}.$$

But the RHS is obviously also a lower bound for

$$\max \left\{ |S(n, t, s, r)| : r = 0, \dots, \frac{n-t-s-1}{2} \right\}.$$

□

5. COMPARISON METHODS AND PROOFS OF THEOREMS 8 AND 10

In this section we work with *size-dependent* weight functions, i.e., functions $\omega : 2^{[n]} \rightarrow \mathbb{R}_+$ for which there are numbers $\omega_0, \dots, \omega_n$ such that $\omega(X) = \omega_i$ for all $X \subseteq [n]$ with $|X| = i$, $i = 0, \dots, n$. We call $\omega := (\omega_0, \dots, \omega_n)$ the *weight vector*.

A corollary of the comparison lemma [2] is the following result proved in [6]:

THEOREM 18. *Let ω be size-dependent. Then*

$$M(I(n, t), \omega) = \omega \left(S \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right)$$

if

$$\max \left\{ \frac{\omega_i}{\omega_{i+1}} : i = t, \dots, n-1 \right\} < 1 + \frac{t-1}{\lfloor \frac{n-t}{2} \rfloor}.$$

REMARK. Using a continuity argument it is easy to see that the relation ‘<’ in the above condition can be replaced by ‘≤’.

In the next lemmas we present conditions for how the weight function can be changed without changing the optimal solution.

LEMMA 19. Let ω be size-dependent and suppose that $M(I(n, t), \omega)$ is attained at $S(n, t, \lfloor \frac{n-t}{2} \rfloor)$. Let ω' be a new size-dependent weight defined by either one of the following assignments:

$$\omega'_i := \begin{cases} \omega_i - \lambda & \text{if } i = u \\ \omega_i + \lambda \frac{\binom{n}{u}}{\binom{n}{\ell}} & \text{if } i = \ell \\ \omega_i & \text{otherwise,} \end{cases} \quad (16)$$

where $0 < \lambda \leq \omega_u$ and, $\frac{n+t}{2} \leq \ell < u \leq n$ or $0 \leq \ell < u < \lfloor \frac{n+t}{2} \rfloor$,

$$\omega'_i := \begin{cases} \omega_i + \delta & \text{if } i = \ell \\ \omega_i & \text{otherwise,} \end{cases} \quad (17)$$

where $\delta > 0$ and $\frac{n+t}{2} \leq \ell \leq n$.

Then $M(I(n, t), \omega')$ is also attained at $S(n, t, \lfloor \frac{n-t}{2} \rfloor)$.

PROOF. Let ω' be given by (16). Note that

$$\omega' \left(S \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right) = \omega \left(S \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right).$$

Let \mathcal{F} be an optimal family for ω' . W.l.o.g. we may assume that \mathcal{F} is a filter (or upset), i.e., $X \in \mathcal{F}$, $X \subseteq Y$ imply $Y \in \mathcal{F}$. By the normalized matching property of the Boolean lattice (cf. [7, p. 149]) we have

$$\frac{|\mathcal{F}_\ell|}{\binom{n}{\ell}} \leq \frac{|\mathcal{F}_u|}{\binom{n}{u}}.$$

It follows that

$$\begin{aligned} \omega'(\mathcal{F}) &= \omega(\mathcal{F}) + \lambda \frac{\binom{n}{u}}{\binom{n}{\ell}} |\mathcal{F}_\ell| - \lambda |\mathcal{F}_u| = \omega(\mathcal{F}) + \lambda \binom{n}{u} \left(\frac{|\mathcal{F}_\ell|}{\binom{n}{\ell}} - \frac{|\mathcal{F}_u|}{\binom{n}{u}} \right) \\ &\leq \omega(\mathcal{F}) \leq \omega \left(S \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right) = \omega' \left(S \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right). \end{aligned}$$

Now let ω' be given by (17) and let \mathcal{F} be an optimal family for ω' . Then

$$\begin{aligned} \omega'(\mathcal{F}) &= \omega(\mathcal{F}) + \delta |\mathcal{F}_\ell| \leq \omega(\mathcal{F}) + \delta \binom{n}{\ell} \\ &\leq \omega \left(S \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right) + \delta \binom{n}{\ell} = \omega' \left(S \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right). \quad \square \end{aligned}$$

LEMMA 20. Let ω be size-dependent and suppose that $M(I(n, t), \omega)$ is attained at $S(n, t, \lfloor \frac{n-t}{2} \rfloor)$. Let $\lambda > 0$, $0 \leq \ell < \lfloor \frac{n+t}{2} \rfloor$ and let ω' be a new size-dependent weight defined by

$$\omega'_i := \begin{cases} \omega_i + \lambda & \text{if } i = \ell \\ \omega_i + \lambda \frac{\ell - t + 1}{\ell} & \text{if } i = n + t - \ell - 1 \\ \omega_i & \text{otherwise.} \end{cases}$$

Then $M(I(n, t), \omega')$ is also attained at $S(n, t, \lfloor \frac{n-t}{2} \rfloor)$.

PROOF. Obviously,

$$\omega' \left(S \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right) = \omega \left(S \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right) + \lambda \frac{\ell-t+1}{\ell} \binom{n}{n+t-\ell-1}.$$

Let \mathcal{F} be an optimal family for ω' . From Katona's theorem concerning shadows of t -intersecting families (cf. [7, p. 301]) it follows that

$$|\mathcal{F}_{n+t-\ell-1}| \leq \binom{n}{n+t-\ell-1} - \frac{\ell}{\ell-t+1} |\mathcal{F}_\ell|.$$

Accordingly,

$$\begin{aligned} \omega'(\mathcal{F}) &= \omega(\mathcal{F}) + \lambda |\mathcal{F}_\ell| + \lambda \frac{\ell-t+1}{\ell} |\mathcal{F}_{n+t-\ell-1}| \\ &\leq \omega(\mathcal{F}) + \lambda \frac{\ell-t+1}{\ell} \left(\frac{\ell}{\ell-t+1} |\mathcal{F}_\ell| + \binom{n}{n+t-\ell-1} - \frac{\ell}{\ell-t+1} |\mathcal{F}_\ell| \right) \\ &\leq \omega \left(S \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right) + \lambda \frac{\ell-t+1}{\ell} \binom{n}{n+t-\ell-1} = \omega' \left(S \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right). \end{aligned}$$

□

PROOF OF THEOREM 8. Obviously, it is enough to prove the assertion for

$$k := \left\lceil \frac{n+t}{2} + \sqrt{\log n \sqrt{n}} \right\rceil$$

(e.g., apply Lemma 19 with (17)). Let

$$q := 1 + \frac{t-1}{\lfloor \frac{n-t}{2} \rfloor}.$$

We consider the size-dependent weight ω defined by

$$\omega_i := \begin{cases} 1 & \text{if } i < \frac{n+t}{2} \\ \frac{1}{q} & \text{if } i \geq \frac{n+t}{2}. \end{cases} \quad (18)$$

By Theorem 18 (and the succeeding remark), we know that $M(I(n, t), \omega)$ is attained at $S(n, t, \lfloor \frac{n-t}{2} \rfloor)$.

Now we apply Lemma 19 with (16) for $\ell = \lceil \frac{n+t}{2} \rceil$ and $u = k+1, k+2, \dots, n$. This gives the new weight vector ω' :

$$\omega'_i := \begin{cases} 1 & \text{if } i < \frac{n+t}{2} \\ \frac{1}{q} \left(1 + \frac{1}{\binom{n}{\lceil (n+t)/2 \rceil}} \sum_{u=k+1}^n \binom{n}{u} \right) & \text{if } i = \left\lceil \frac{n+t}{2} \right\rceil \\ \frac{1}{q} & \text{if } \left\lceil \frac{n+t}{2} \right\rceil < i \leq k \\ 0 & \text{if } i > k. \end{cases}$$

It is known (cf. [12, p. 284]) that, as $n \rightarrow \infty$,

$$\binom{n}{\lceil (n+t)/2 \rceil} \sim \frac{2^{n+1}}{\sqrt{2\pi n}}, \quad (19)$$

and, with $x = o(n^{\frac{1}{6}})$, $x \rightarrow \infty$,

$$\sum_{u > \frac{n}{2} + x \frac{\sqrt{n}}{2}} \binom{n}{u} \sim \frac{1}{\sqrt{2\pi x}} e^{-x^2/2} 2^n.$$

The last formula with $x = 2\sqrt{\log n}$ implies

$$\sum_{u=k+1}^n \binom{n}{u} \lesssim \frac{1}{2\sqrt{\pi\sqrt{\log n}}} \frac{2^n}{n^2}. \quad (20)$$

By (19) and (20) we have for sufficiently large n

$$\frac{1}{\binom{n}{\lceil (n+t)/2 \rceil}} \sum_{u=k+1}^n \binom{n}{u} < \frac{t-1}{\lfloor \frac{n-t}{2} \rfloor} = q-1$$

which implies that $\omega'_i \leq 1$ for $i = \lceil \frac{n+t}{2} \rceil, \dots, k$. Hence, by again applying Lemma 8 with (17) we obtain that for large n

$$M(I_{\leq k}(n, t)) = \left| S_{\leq k} \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right|. \quad \square$$

PROOF OF THEOREM 10. We use the same method as in the proof of Theorem 8, but here we put

$$k := \left\lceil \frac{n+t}{2} \right\rceil + c,$$

where c is an integer. Recalling (18) we have to show that there exists c such that for large n

$$\frac{1}{q} \left(1 + \frac{1}{\binom{n}{\lceil (n+t)/2 \rceil}} \sum_{u=k+1}^n \binom{n}{u} \right) \leq 1,$$

or, equivalently,

$$\sum_{u=k+1}^n \binom{n}{u} \leq (q-1) \binom{n}{\lceil (n+t)/2 \rceil}. \quad (21)$$

We have

$$\frac{1}{q} > \frac{\binom{n}{\lceil (n+t)/2 \rceil + 1}}{\binom{n}{\lceil (n+t)/2 \rceil}} > \dots > \frac{\binom{n}{n}}{\binom{n}{n-1}}$$

and consequently

$$\sum_{u=k+1}^n \binom{n}{u} < \binom{n}{\lceil (n+t)/2 \rceil} \sum_{u=k+1}^n q^{-(u - \lceil \frac{n+t}{2} \rceil)} < \binom{n}{\lceil (n+t)/2 \rceil} q^{-(c+1)} \frac{1}{1 - q^{-1}}.$$

Therefore,

$$\frac{(1/q)^{c+1}}{1 - 1/q} \leq q-1,$$

or, equivalently,

$$q^c \geq \frac{1}{(q-1)^2} \quad (22)$$

is sufficient for (21). Using

$$q^c \geq c(q-1)$$

we see that

$$c \geq \frac{1}{(q-1)^3}$$

is sufficient for (22). However, for $t \geq \delta n$, the last condition certainly holds (for large n) if

$$c > \left(\frac{1-\delta}{2\delta} \right)^3. \quad \square$$

6. PROOF OF THEOREM 12

LEMMA 21. *Let*

$$a_{k,n} = \frac{1}{\binom{n}{k}} \sum_{j=0}^k \binom{n}{j}.$$

Then $a_{k,n}$ is increasing in k (for $k = 0, \dots, n$).

PROOF. For fixed n we have $a_{k,n} \leq a_{k+1,n}$ iff

$$\binom{n}{k} \sum_{j=0}^{k+1} \binom{n}{j} - \binom{n}{k+1} \sum_{j=0}^k \binom{n}{j} \geq 0.$$

However, this inequality is true since the LHS is not less than

$$\sum_{j=0}^k \left(\binom{n}{k} \binom{n}{j+1} - \binom{n}{k+1} \binom{n}{j} \right)$$

and each term of the last sum is nonnegative by the log-concavity of the binomial coefficients. \square

LEMMA 22. *Let $k < \frac{n+t}{2}$. Then the sequence*

$$|S_{\leq k}(n, t, 0)|, |S_{\leq k}(n, t, 1)|, \dots, \left| S_{\leq k} \left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor \right) \right|$$

is unimodal.

PROOF. By considering $|S_{\leq k}(n, t, r) \setminus S_{\leq k}(n, t, r+1)|$ and $|S_{\leq k}(n, t, r+1) \setminus S_{\leq k}(n, t, r)|$ we see that

$$|S_{\leq k}(n, t, r)| \leq |S_{\leq k}(n, t, r+1)|$$

is equivalent to

$$(t+r) \binom{n-t-2r-2}{k-t-r} \leq (t-1) \sum_{i=0}^{k-t-r} \binom{n-t-2r-2}{i}. \quad (23)$$

We will show that $|S_{\leq k}(n, t, r)| \leq |S_{\leq k}(n, t, r+1)|$ implies $|S_{\leq k}(n, t, r-1)| \leq |S_{\leq k}(n, t, r)|$. It suffices to prove that for all r with $0 < r < \lfloor \frac{n-t}{2} \rfloor$

$$\binom{n-t-2r}{k-t-r+1} \sum_{i=0}^{k-t-r} \binom{n-t-2r-2}{i} \leq \binom{n-t-2r-2}{k-t-r} \sum_{i=0}^{k-t-r+1} \binom{n-t-2r}{i},$$

or, (substituting $a = n - t - 2r - 2$, $b = k - t - r$) that for all a, b with $2b < a + 2$

$$\binom{a+2}{b+1} \sum_{i=0}^b \binom{a}{i} \leq \binom{a}{b} \sum_{i=0}^{b+1} \binom{a+2}{i}.$$

Subtracting

$$2 \binom{a}{b} \sum_{i=0}^b \binom{a}{i}$$

from the last inequality gives

$$\left(\binom{a}{b-1} + \binom{a}{b+1} \right) + \sum_{i=0}^b \binom{a}{i} \leq \binom{a}{b} \left(\sum_{i=0}^{b+1} \binom{a}{i} + \sum_{i=0}^{b-1} \binom{a}{i} \right). \quad (24)$$

Using $2b \leq a + 1$ one verifies easily that for $i = 0, 1, \dots, b$

$$\frac{\binom{a}{b-1} + \binom{a}{b+1}}{\binom{a}{b}} \leq \frac{\binom{a}{i-1} + \binom{a}{i+1}}{\binom{a}{i}}$$

from which (24) follows. \square

Proof of Theorem 12

Step 1. Let the weight vector ω be defined by

$$\omega_i := \begin{cases} 1 & \text{if } i \leq k \\ 0 & \text{if } i > k. \end{cases}$$

Let $r^* = r^*(n, k)$ be the least r such that

$$|\omega(S(n, t, r))| \geq |\omega(S(n, t, r+1))| \geq \dots \quad (25)$$

By Lemma 22 we know that $|S_{\leq k}(n, t, r^*)| = \max\{|S_{\leq k}(n, t, r)| : r = 0, \dots, \lfloor \frac{n-t}{2} \rfloor\}$. In addition, we have

$$\omega_i = 0 \quad \text{if } i \geq \frac{n+t}{2}. \quad (26)$$

Given an arbitrary weight vector satisfying (25) and (26) it follows by the method of generating sets [1] that

$$M(I(n, t), \omega) = M(I(t + 2r^*, t), \omega'),$$

where the weight vector ω' is given by

$$\omega'_i := \sum_{j=0}^{n-t-2r^*} \omega_{i+j} \binom{n-t-2r^*}{j}$$

for $i = 0, \dots, t + 2r^*$ (cf. [6, Theorem 15 and Example 4]). Hence, in our case, we have

$$M(I_{\leq k}(n, t)) = M(I(t + 2r^*, t), \omega'),$$

where

$$\omega'_i = \sum_{j=0}^{k-i} \binom{n-t-2r^*}{j}$$

for $i = 0, \dots, t + 2r^*$.

Step 2. From Step 1 we know that there is an optimal family \mathcal{F} (i.e., $\mathcal{F} \in I_{\leq k}(n, t)$, $|\mathcal{F}| = M(I_{\leq k}(n, t))$) which has the following property:

$$X \in \mathcal{F} \text{ implies } Y \in \mathcal{F} \text{ for all } Y \in \binom{[n]}{\leq k} \text{ with } Y \cap [t + 2r^*] = X \cap [t + 2r^*]. \quad (27)$$

W.l.o.g. we assume that \mathcal{F} is *left-compressed*, i.e., $(X \setminus \{i\}) \cup \{j\} \in \mathcal{F}$ for all $i, j \in [n]$ with $i > j$, $i \in X$, $j \notin X$. We will prove by pushing–pulling [3] that \mathcal{F} is invariant in $[t + 2r^*]$, i.e., $(X \setminus \{i\}) \cup \{j\} \in \mathcal{F}$ for all $i, j \in [t + 2r^*]$, $i \in X$, $j \notin X$. Assume the contrary. Let

$$\begin{aligned} \ell &= \max\{i : \mathcal{F} \text{ is invariant in } [i]\} \\ \mathcal{L} &= \{X \in \mathcal{F} : \ell + 1 \notin X, (X \setminus \{i\}) \cup \{\ell + 1\} \notin \mathcal{F} \text{ for some } i \in X \cap [\ell]\} \\ \mathcal{L}^* &= \{X \cap [\ell + 2, n] : X \in \mathcal{L}\}. \end{aligned}$$

Furthermore, let $\mathcal{L}_i = \{X \in \mathcal{L} : |X \cap [\ell] = i\}$, $\mathcal{L}_i^* = \{X \cap [\ell + 2, n] : X \in \mathcal{L}_i\}$. By our assumption we have $\ell < t + 2r^*$. The following facts follow from the pushing–pulling method (cf. [6]):

- (i) \mathcal{L} is nonempty and invariant in $[\ell]$.
- (ii) $\ell \geq t$, $2 \mid \ell + t$, $\mathcal{L}_i = \emptyset$ for $i \in [\ell] \setminus \{\frac{\ell+t}{2}\}$.
- (iii) For all intersecting subfamilies \mathcal{T}^* of $\mathcal{L}_{\frac{\ell+t}{2}}^*$,

$$\frac{\sum_{X \in \mathcal{T}^*} \omega_{|X| + \frac{\ell+t}{2}}}{\sum_{X \in \mathcal{L}_{\frac{\ell+t}{2}}^*} \omega_{|X| + \frac{\ell+t}{2}}} \leq \frac{\ell - t + 2}{2(\ell + 1)}.$$

It is easy to see that $\ell = t + 2r^* - 2$ is impossible (e.g., since $\mathcal{L} \neq \emptyset$ we have $t + 2r^* \notin X$ for some $X \in \mathcal{L}_{\frac{\ell+t}{2}}^*$ which implies $\mathcal{F} = S_{\leq k}(n, t, r^* - 1)$ in contradiction to the choice of \mathcal{F} and r^*). Hence $\ell \leq t + 2r^* - 4$. We show that the family $\mathcal{T}^* = \{X \in \mathcal{L}_{\frac{\ell+t}{2}}^* : n \in X\}$ contradicts fact (iii). Indeed, recalling (27), this will follow from the next inequality (we classify the members X of $\mathcal{L}_{\frac{\ell+t}{2}}^*$ and \mathcal{T}^* with respect to $i = |X \cap [\ell + 2, t + 2r^*]|$).

CLAIM. *If $k \leq (\frac{1}{2} - \epsilon)n$ and n is sufficiently large then we have for all ℓ, i with $\ell \leq t + 2r^* - 4$, $2 \mid \ell + t$, $0 \leq i \leq t + 2r^* - \ell - 1$*

$$\sum_{j=0}^{k - \frac{\ell+t}{2} - i - 1} \binom{n - t - 2r^* - 1}{j} > \frac{\ell - t + 2}{2(\ell + 1)} \sum_{j=0}^{k - \frac{\ell+t}{2} - i} \binom{n - t - 2r^*}{j}.$$

This inequality is easily seen to be equivalent to

$$\frac{n - t - 2r^*}{n - t - 2r^* - k + \frac{\ell+t}{2} + i} \frac{\sum_{j=0}^{k - \frac{\ell+t}{2} - i} \binom{n - t - 2r^*}{j}}{\binom{n - t - 2r^*}{k - \frac{\ell+t}{2} - i}} > \frac{\ell + 1}{t - 1}. \quad (28)$$

Since $\ell \leq t + 2r^* - 4$ it suffices to show that the LHS of (28) is greater than

$$\frac{t + 2r^* - 3}{t - 1}.$$

For every r let

$$\kappa_r = \frac{r}{t + 2r - 1} \quad \text{and} \quad m_r = \frac{\kappa_{r-1} + \kappa_r}{2}.$$

Note that $r = (t-1)\frac{\kappa_r}{1-2\kappa_r}$ and that κ_r is strictly increasing and $\lim_{r \rightarrow \infty} \kappa_r = \frac{1}{2}$. We consider the finite set

$$R := \{r \in \mathbb{N} : \kappa_r \leq \frac{1}{2} - \epsilon\}.$$

Since for $\kappa < \frac{1}{2}$, $c \in \mathbb{N}$ constant

$$\lim_{n \rightarrow \infty} \frac{1}{\binom{n}{\lfloor \kappa n \rfloor + c}} \sum_{j=0}^{\lfloor \kappa n \rfloor + c} \binom{n}{j} = \frac{1-\kappa}{1-2\kappa}$$

(cf. [5]), we have for sufficiently large n and all r, ℓ, i with $r \in R$, $\ell \leq t+2r-4$, $2 \mid \ell+t$, $0 \leq i \leq t+2r-\ell-1$

$$\frac{n-t-2r}{n-t-2r-\lfloor m_r n \rfloor + \frac{\ell+t}{2} + i} \frac{\sum_{j=0}^{\lfloor m_r n \rfloor - \frac{\ell+t}{2} - i} \binom{n-t-2r}{j}}{\binom{n-t-2r}{\lfloor m_r n \rfloor - \frac{\ell+t}{2} - i}} > \frac{1}{1-2\kappa_{r-1}} = \frac{t+2r-3}{t-1}. \quad (29)$$

Analogously, we have for sufficiently large n and all $r \in R$

$$\frac{\sum_{j=0}^{\lfloor m_{r+1} n \rfloor - t - r} \binom{n-t-2r-2}{j}}{\binom{n-t-2r-2}{\lfloor m_{r+1} n \rfloor - t - r}} < \frac{1-\kappa_{r+1}}{1-2\kappa_{r+1}} = \frac{t+r}{t-1}. \quad (30)$$

Now let n be such that (29) and (30) are satisfied and let r be determined by

$$\lfloor m_r n \rfloor \leq k < \lfloor m_{r+1} n \rfloor.$$

By (23), Lemma 21 and (30) we have

$$|S_{\leq k}(n, t, r)| > |S_{\leq k}(n, t, r+1)|,$$

hence, by Lemma 22, $r^* \leq r$. Lemma 21 and (29) now imply that (28) is satisfied. \square

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